The Theory of Dimensioned Matrices

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Abstract

Dimensioned matrices are defined as rectangular arrays of elements from a typed family of fields (TFF). The TFF has elements which are ordered pairs, (f, g), $f \in F$, $g \in G$, where F is a field and G is a group of "types." Operations are defined so that the TFF is closed under multiplication, but not addition. The TFF is a model of physically dimensioned scalars, e.g., 1 meter, 2 seconds, where f represents the numeric component, and g the dimension, and products are always defined, but only sums of the same type are defined. Dimensioned vectors, i.e., $n \times 1$ dimensioned matrices, contain n TFF elements (modeling multidimensional signals or states,) and are operated upon by multiplication with dimensioned matrices. Their algebra has properties which emerge as surprising from the standpoint of traditional linear algebra. This paper describes the structure of several important classes of dimensioned matrices which have significant implications for applied mathematics and multidimensional systems theory.

1 Introduction

Physically dimensioned scalar quantities, e.g., '3 meters,' can be modelled as ordered pairs of a field element and a "dimension." Dimensions have a group structure, with:

i)	a binary operation,	e.g., "area" $*$ "length" = "volume"
ii)	an identity,	e.g., "area" $*$ "dimensionless" = "area"
iii)	inverses,	e.g., "area" $*$ "1/area" = "dimensionless"

References [1–9] give various algebraic models for dimensioned scalars. Usually g is taken as a vector of exponents on fundamental units of length, mass, time, charge, etc., and the group operation is vector addition of these exponents. It is well established that all physically meaningful scalar calculations can be explicitly carried out with dimensioned quantities in such a way that all equations and all sums are dimensionally homogeneous.

This paper develops the theory of vectors and matrices which contain such dimensioned scalars as their components. As motivation, consider these 2×2 arrays,

$$(1) \qquad X = \begin{bmatrix} 1\mathrm{m} & 1\mathrm{s} \\ 1\mathrm{s} & 1\mathrm{m} \end{bmatrix}, \qquad Y = \begin{bmatrix} 1\mathrm{m} & 1\mathrm{ms} \\ 1\mathrm{m/s} & 1\mathrm{m} \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 1\mathrm{s} \\ 1/\mathrm{s} & 1 \end{bmatrix},$$

where m abbreviates meters, and s, seconds. Attempting to directly multiply out matrix products, using the standard formalism, $[AB]_{ij} = \sum_k A_{ik} B_{kj}$, shows that the product X^2 is undefined, as its diagonal elements would be the undefined sum $1\mathrm{m}^2 + 1\mathrm{s}^2$. Thus the familiar property that "any square matrix can be squared" does not hold in this algebra, as matrix products contain scalar sums, and sums are only sometimes defined. If matrix elements are

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chosen carefully however, products are defined, e.g., direct multiplication shows that:

$$Y^2 = \begin{bmatrix} 2\mathrm{m}^2 & 2\mathrm{m}^2\mathrm{s} \\ 2\mathrm{m}^2/\mathrm{s} & 2\mathrm{m}^2 \end{bmatrix}, \qquad Z^2 = \begin{bmatrix} 2 & 2\mathrm{s} \\ 2/\mathrm{s} & 2 \end{bmatrix}. \tag{2}$$

An important difference between Y and Z is that Z^2 preserves the dimensions of Z, so that $Y + Y^2$ is undefined, but $Z + Z^2$ is defined. As a consequence of this property, all integer powers of Z have the same dimensional structure to the components, and so polynomial functions of Z are defined. In particular, the Taylor series for the matrix exponential requires summing powers, so e^Z is defined, but e^X and e^Y are undefined.

These different properties show that X, Y, and Z come from three different classes of dimensioned matrices. This paper sorts out the properties of these and other dimensioned matrix classes. Additional classes, details, and analysis can be found in [3].

2 Definitions

A *typed family of fields* (TFF) over a field F and a group G is the Cartesian product $F \times G$, along with two operations defined below. A typical element is written (f, g), and g is called its *type*. Let T be the *type function*, defined by $T((f, g)) = g$. A product operation is defined in the TFF as

$$(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_1 g_2), \tag{3}$$

where $f_1 f_2$ indicates the product in F, and $g_1 g_2$ denotes the group operation in G.

An addition operation in the TFF is only defined for elements of the same type:

$$(f_1, g_1) + (f_2, g_2) = \begin{cases} (f_1 + f_2, g_1), & \text{if } g_1 = g_2; \\ \text{undefined}, & \text{otherwise.} \end{cases} \tag{4}$$

An element c is defined to be *dimensionless* iff $T(c) = 1$, i.e., c is of the form $(f, 1)$, where 1 is the identity in G. A *zero* of the TFF is an element (f, g) for which $f = 0$ in F. It is immediate that $(0, g)$ is a unique additive identity element for all elements of type g, that $(1, 1)$ is the unique multiplicative identity element for the entire TFF, and that the operations in the TFF obey the usual commutative, associative, and distributive axioms of fields, whenever the sums are defined, except that multiplication is only commutative if G is Abelian. Elements of the TFF are called *dimensioned scalars.*

An $n \times m$ *dimensioned matrix* (DM) is an array of dimensioned scalars. It is called *dimensionless* iff all of its elements are dimensionless. A *dimensioned vector* is an $n \times 1$ dimensioned matrix. Matrices will be written $A, B, \ldots$; vectors, x, y; and scalars, a, b, c. The ij^{th} element of A is written A_{ij} or $[A]_{ij}$, and the i^{th} element of x is written x_i. A is called uniform iff $\forall i, j,\ T(A_{i,j}) = T(A_{1,1})$. The transpose, A^T, satisfies $[A^T]_{ij} = A_{ji}$.

If A and B are both $n \times m$ dimensioned matrices, their sum is defined componentwise in the usual manner,

$$[A + B]_{ij} = A_{ij} + B_{ij}, \quad \text{for } i = 1 \text{ to } n, \text{ and } j = 1 \text{ to } m, \tag{5}$$

except that the sum is undefined if any of the nm scalar sums are undefined. If A is $n \times m$ and B is $m \times p$, their product is defined componentwise in the usual manner,

$$[AB]_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}, \quad \text{for } i = 1 \text{ to } n, \text{ and } j = 1 \text{ to } p, \tag{6}$$

except that the product is undefined if any of the sums are undefined. The scalar product of a scalar c and any matrix A is defined in the usual manner as a matrix of the same size obtained by bringing c into every component of A.

$$[cA]_{ij} = [Ac]_{ij} = cA_{ij}, \quad \text{for } i = 1 \text{ to } n, \text{ and } j = 1 \text{ to } m. \tag{7}$$

An equivalence relation of *dimensional similarity* is defined on matrices, written $A \sim B$. If A and B are both $n \times m$ dimensioned matrices, define

$$A \sim B \qquad \text{iff} \qquad \forall i, j,\ T(A_{ij}) = T(B_{ij}) \tag{8}$$

A weaker equivalence relation of *dimensional parallelism* is also defined on matrices, written $A \approx B$. If A and B are both $n \times m$ dimensioned matrices, define

$$A \approx B \qquad \text{iff} \qquad \exists c,\ A \sim cB \tag{9}$$

Examples, using the matrices of (1) and (2) are: $Z \sim Z^2$, $Y \not\sim Y^2$, $Y \approx Y^2 \approx Z \approx Z^2$, and $X \not\approx Y$. As c in (9) can be dimensionless, $A \sim B \Rightarrow A \approx B$.

A *dimensional inverse* of an $n \times m$ matrix A, written $A^{\sim}$, is defined to be any $m \times n$ matrix which can be obtained by transposing A, and replacing its elements with elements of the inverse types. Specifically,

$$\forall i, j,\ T([A^{\sim}]_{ij}) = \big(T([A]_{ji})\big)^{-1}, \tag{10}$$

where the inverse is in G. No constraints are placed on the F components of the elements in $A^{\sim}$, so the dimensional inverse will only appear within relations of dimensional similarity or dimensional parallelism, not equality. As examples, $Z \sim Z^{\sim}$, $Y \not\sim Y^{\sim}$, and $Y \approx Y^{\sim}$.

For x, a dimensioned vector, the *complete dimensioned vector space* (CDVS) of type x is the set of y such that $y \sim x$. A *dimensioned vectorspace* (DVS) of type x is a subset of a CDVS which is closed under addition and scalar multiplication by dimensionless scalars. A DVS of type x is called an *x-space*.

3 Results

Some key ideas of "dimensioned linear algebra" can now be given.

3.1 Basic Properties

The following basic properties are straightforwardly derived by applying definitions (3)–(10), so proofs are not given here. (See [3].) If an operation is undefined, that is because it entails a scalar sum of different types, and so is undefined by (4).

$$A^{\sim\sim} \sim A \tag{11}$$
$$(A^{\sim})^T \sim (A^T)^{\sim} \tag{12}$$
$$(AB)^{\sim} \sim B^{\sim}A^{\sim} \tag{13}$$
$$[xy^T]_{ij} \sim x_i y_j \tag{14}$$
$$[xy^{\sim}]_{ij} \sim x_i / y_j \tag{15}$$
$$x^{\sim}x \text{ is always defined, and } T(x^{\sim}x) = 1 \tag{16}$$
$$x^T x \text{ is defined} \iff x \text{ is uniform} \tag{17}$$
$$x^{\sim}y \text{ is defined} \iff x \approx y \tag{18}$$
$$x^T y \text{ is defined} \iff x \approx y^{\sim T} \tag{19}$$

Outer products (14,15) are always defined. Inner products (16)–(19) involve sums, so are only defined in special cases. From (17), magnitudes can only be defined on a space of uniform vectors, so familiar derivations involving magnitudes are not general to all x-spaces.

3.2 Useful Matrices are Dimensionally of Rank 1

If the product Ax is defined, an inner product is defined between each row of A and x. From (19) and the transitivity of '$\approx$', it follows that all the rows of A are dimensionally parallel to each other. Thus the dimensional form of A can be factored as an outer product

$$A \sim \begin{bmatrix} 1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} [\text{first row of } A].$$

So, all the columns are dimensionally parallel as well. This, and a dual argument for products $x^T A$, leads to the following theorem, which also applies to matrix-matrix products.

THEOREM 3.1. *A can appear in at least one product $\iff$ $\exists x, y,$ such that $A \sim yx^{\sim}$.*
For example, X of (1) can enter into no products and can not be factored as $X \sim yx^{\sim}$.

If A is a matrix representing a linear transformation from an x-space to a y-space, so that $y = Ax$, then the dimensional form of A is determined by x and y as follows. Using (16) and associativity, if $A \sim yx^{\sim}$, then $Ax \sim (yx^{\sim})x \sim y(x^{\sim}x) \sim y$, since a dimensionless scalar does not affect the result. This results in the following:

THEOREM 3.2. *The matrix of a linear transformation from an x-space to a y-space has the form $yx^{\sim}$.*

As a consequence, a mapping in the reverse direction of $A \sim yx^{\sim}$ is provided by $A^{\sim} \sim xy^{\sim}$. Thus an adjoint of a dimensioned matrix A is of the form $A^{\sim}$, not A^T.

3.3 Endomorphic Matrices and the Exponential

Matrices of the form $xx^{\sim}$ are termed *endomorphic*, as they can be used as a transformation matrix for a linear transformation from an x-space into itself. Because $(xx^{\sim})(xx^{\sim}) \sim x(x^{\sim}x)x^{\sim} \sim xx^{\sim}$, endomorphic matrices reproduce their dimensions when squared, e.g., Z in (1). So, the exponential and other Taylor-series based transcendental functions are defined on endomorphic matrices. This gives:

THEOREM 3.3. *$\exists x,\ A \sim xx^{\sim}$ $\iff$ $A^n \sim A$ $\iff$ e^A is defined*

From (15), the diagonal of an endomorphic matrix is dimensionless, which implies the familiar, special, 1×1 case: a scalar argument to a transcendental function must be dimensionless.

Identity matrices are clearly endomorphic. There are many different $n \times n$ identity matrices with different zeros, but not as many matrices as spaces, since $x \approx y \Rightarrow xx^{\sim} \sim yy^{\sim}$.

3.4 Squarable Matrices and Eigenstructure

Define a matrix A to be *squarable* iff for some c and x, A is of the form $cxx^{\sim}$. Then $A^2 \sim (cxx^{\sim})(cxx^{\sim}) \sim c^2x(x^{\sim}x)x^{\sim} \sim c^2xx^{\sim} \sim cA$, so such matrices are indeed squarable. Conversely, using (18), let $A \sim yx^{\sim}$, so if $A^2 \sim yx^{\sim}yx^{\sim}$ is defined, then the existence of the inner term $x^{\sim}y$ ensures $x \approx y$, so A can be expressed as $cxx^{\sim}$. Thus:

THEOREM 3.4. *A^2 is defined $\iff$ $\exists c, x,\ A \sim cxx^{\sim}$.*

Y in (1) is squarable but not endomorphic. Define eigenvectors and eigenvalues of dimensioned matrices, as usual, as solutions to $Ax = \lambda x$. Then from Theorem 3.2, $A \sim \lambda xx^{\sim}$, a squarable matrix. The remaining details of the following theorem are easy to fill in.

THEOREM 3.5. *A has eigenstructure iff A is squarable. If λ is an eigenvalue and y is an eigenvector of $A \sim cxx^{\sim}$, then $\lambda \sim c$ and $y \approx x$.*

Most square matrices are not squarable and so have no eigenstructure. In particular, symmetric matrices (which have the form xx^T) are generally not squarable, so the analysis of positive definite matrices in terms of eigenstructure is not generally meaningful.

3.5 Additional Results

Listed here are a few other results. For more, plus derivations and discussion, see [3].

(20) $$A \sim yx^{\sim} \text{ is } n \times n \quad \Longrightarrow \quad \det(A) \text{ is defined and } \det(A) \sim \prod_{i=1}^{n} \frac{y_i}{x_i}$$

(21) $$A \sim yx^{\sim} \text{ is nonsingular} \quad \Longrightarrow \quad A^{-1} \sim A^{\sim} \sim xy^{\sim}$$

(22) $$x^T A x \text{ is defined} \quad \Longleftrightarrow \quad A \approx x^{\sim T} x^{\sim}$$

(23) $$AA^{-1} = A^{-1}A \quad \Longleftrightarrow \quad A \text{ is uniform}$$

(24) $$A \sim cxx^{\sim} \text{ and } A \sim A^T \quad \Longrightarrow \quad A \text{ is uniform}$$

(25) $$U^T = U^{-1} \quad \Longrightarrow \quad U \text{ is dimensionless}$$

(26) $$A = U\Sigma V^T \text{ is a SVD decomposition} \quad \Longrightarrow \quad A \text{ is uniform}$$

As a consequence of (24), the eigenstructure of a symmetric matrix is only defined if the matrix is uniform. Orthogonal matrices are dimensionless (25), so the singular value decomposition factorization $U\Sigma V$ can only give a uniform matrix.

4 Conclusions

Different classes of dimensioned matrices have different operations defined upon them. Attempting to apply operations to classes for which the operation is not defined entails adding scalars of different types, which is a physically meaningless numeric manipulation. Important results of this framework with consequences to multidimensional systems theory are: (1) A^T is not the adjoint of A; (2) e^A is defined iff A is endomorphic; (3) Symmetric matrices generally have no eigenstructure; and (4) the SVD is only defined on uniform matrices. Using traditional (dimensionless) linear algebra in these situations, on numeric matrices obtained by ignoring the tacit dimensional components, may lead to misleading and physically meaningless results. For further details and discussion, see [3].

References

[1] Brand, L., *The Pi Theorem of Dimensional Analysis,* Arch. Rational Mechanics Anal. 1, (1957), pp. 35–45.

[2] Drobot, S., *On the Foundations of Dimensional Analysis,* Studia Mathematica 14, (1954), pp. 84–99.

[3] Hart, G.W., *Multidimensional Analysis,* to be published by Springer Verlag, (1994).

[4] Kasprzak, W., B. Lysik, and M. Rybaczuk, *Dimensional Analysis in the Identification of Mathematical Models,* World Scientific, Singapore, (1990).

[5] Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky, *Foundations of Measurement, Vol. 1: Additive and Polynomial Representations,* Academic Press, New York, (1971).

[6] Kurth, R., *A Note on Dimensional Analysis,* American Mathematical Monthly 72, (1965), pp. 965–969.

[7] Quade, W., *The Algebraic Structure of Dimensonal Analysis,* appendix to F.J. de Jong, *Dimensional Analysis for Economists,* North Holland, (1967), pp. 143–199.

[8] Thun, R.E., *On Dimensional Analysis,* IBM J. Research and Development 4, (1960), pp. 349–356.

[9] Whitney, H., *The Mathematics of Physical Quantities. I: Mathematical Models for Measurement. II: Quantity Structures and Dimensional Analysis,* American Mathematical Monthly 75, (1968), pp. 115–138 and 227–256.